Quantum phase transition of dynamical resistance in a mesoscopic capacitor

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Abstract. We study theoretically dynamic response of a mesoscopic capacitor, which consists of a quantum dot connected to an electron reservoir via a point contact and capacitively coupled to a gate voltage. A quantum Hall edge state with a filling factor ν is realized in a strong magnetic field applied perpendicular to the two-dimensional electron gas. We discuss a noise-driven quantum phase transition of the transport property of the edge state by taking into account an ohmic bath connected to the gate voltage. Without the noise, the charge relaxation for $\nu > 1/2$ is universally quantized at $R_q = h/(2e^2\nu)$, while for $\nu < 1/2$, the system undergoes the Kosterlitz-Thouless transtion, which drastically changes the nature of the dynamical resistance. The phase transition is facilitated by the noisy gate voltage, and we see that it can occur even for an integer quantum Hall edge at $\nu = 1$. When the dissipation by the noise is sufficiently small, the quantized value of R_q is shifted by the bath impedance.

1. Introduction

In the long history of the mesoscopic physics, it is quite recent that it became possible to observe dynamic response of coherent systems in experiments [1, 2]. Theoretically, dynamical transport properties of the mesoscopic systems has been extensively investigated with the scattering theory [3, 4]. One of the most intriguing results of the theoretical efforts is the universal quantization of charge relaxation resistance in mesoscopic capacitors [4]. The mesoscopic
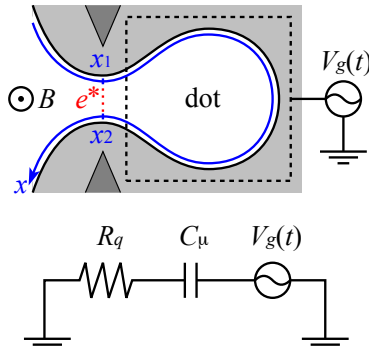

Figure 1. Schematic view of the mesoscopic capacitor and its equivalent RC circuit. The quantum dot is connected to an electron reservoir via a point contact and capacitively coupled to a gate (shown as a dotted square) with time-dependent voltage $V_g(t)$. A magnetic field applied perpendicular to the plane realizes a quantum Hall edge state moving along the x axis. The dotted line between x_1 and x_2 represents quasiparticle tunneling on the point contact.

capacitors are quantum analogs of classical RC circuits, among which the simplest one (Fig. 1) consists of a quantum dot connected to an electron reservoir via a quantum point contact and capacitively coupled to an ac gate voltage. An electron entering the quantum dot relaxes to equilibrium in the scale of charge relaxation time τ_{RC}. The dynamical resistance observed at this time scale is the so-called *charge relaxation resistance.* It has been predicted by Büttiker *et al.* that the charge relaxation resistance is quantized at half a resistance quantum $h/(2e^2)$ irrespective of the property of the point contact [4]. Indeed, the universal behavior has been confirmed experimentally [1], which has further attracted the theorists' interest in the charge relaxation resistance [5, 6, 7]. Recently, we have investigated theoretically the charge relaxation resistance, treating electron interactions exactly [6]. We have considered a one-dimensional mesoscopic capacitor and predicted the followings; the charging effect of the quantum dot has no influence on the quantization, while short-range interactions described by the Luttinger parameter K drastically affects the nature of the dynamical resistance. In contrast to the universal quantization at $R_q = h/(2e^2 K)$ for $K > 1/2$, the Kosterlitz-Thouless transition occurs for $K < 1/2$ and as a result one can not any more observe the universal charge relaxation resistance. In this paper, we extend our theory to discuss the quantum phase transition driven by a noisy gate voltage. How our treatment of the charging effect is related to the conventional picture of the scattering theory is also discussed by applying the mean field approximation.

2. Model

The system we have in mind is the mesoscopic capacitor depicted in Fig 1. Due to the magnetic field applied perpendicular to the 2DEG, a chiral mode called quantum Hall edge state is realized along the x axis. It has been shown by Wen [8] that the edge state at filling factor ν ($1/\nu$ is an odd integer) is a chiral Luttinger liquid, and its Hamiltonian reads

$$H_0 = \frac{v}{4\pi\nu} \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\partial\phi}{\partial x}\right)^2. \tag{1}$$

Here the bosonic field ϕ describes the gapless edge excitations, whose velocity is denoted by v. The narrow constriction on the point contact causes intra-edge tunneling of Laughlin quasiparticles between $x = x_1$ and $x = x_2$. Since a Laughlin quasiparticle at x is annihilated by the operator $\propto \exp[i\phi(x)]$, the tunneling process is written as

$$H_V = V \cos[\phi(x_1) - \phi(x_2)], \tag{2}$$

where V is the strength of quasiparticle tunneling [1]. Change in the dot charge Q in the range of $x_1 \leq x \leq x_2$ is strongly restricted by the long-range interaction between electrons, i.e., the charging effect. Upon using the fact that the electron density is bosonized as $\rho = \partial_x\phi/(2\pi)$, one can see that the term describing the charging effect takes the form

$$H_C = \frac{Q^2}{2C} + QV_g(t) \qquad \left(Q = \frac{e}{2\pi}[\phi(x_2) - \phi(x_1)]\right) \tag{3}$$

where C is the geometrical capacitance essentially determined by the charging effect. When the gate voltage is oscillating at a low frequency $\omega \ll 1/\tau_{RC}$, the admittance of the quantum RC circuit can be expanded as $G(\omega) = -i\omega C_\mu + \omega^2 C_\mu{}^2 R_q + \mathcal{O}(\omega^3)$, from which the charge relaxation resistance R_q is obtained. Note that the electrochemical capacitance defined by $C_\mu \equiv -\partial\langle Q\rangle/\partial V_g$ in general differs from C, since, e.g., the quasiparticle tunneling induces Coulomb blockade oscillation in C_μ. The above expansion of $G(\omega)$ is possible only when the τ_{RC} is finite; if the phase transition occurs, τ_{RC} diverges faster than $1/T$ as $T \to 0$, so that one can not even regard R_q as the charge relaxation resistance [6].

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[1] In the argument of the cosine in Eq. (2), we have omitted the phase that an edge mode gains along the circumference of the dot by shifting zero of the field ϕ.

3. Mean field theory

At a glance, our formalism described in the previous section looks slightly different from the scattering theory [3, 4], where the charging effect is incorporated through an effective gate voltage U_g in the quantum dot. Clearly, the latter treatment is justified when fluctuation of the dot charge Q is negligible enough that one can approximate the interaction of an electron with the others in the dot with an effective potential created by mean charge $\langle Q \rangle$. Thus one can expect that the two-capacitor obtained from the scattering theory [4] is reproduced by applying the mean field theory to our model. In this section, we first confirm this and compare the results of the scattering theory and our calculation where the interaction is exactly treated. Upon neglecting higher orders in $Q - \langle Q \rangle$ in the charging term (3), we obtain

$$H_C^{\mathrm{MF}} \simeq \frac{\langle Q \rangle}{C} + Q V_g \equiv Q U_g, \qquad \therefore \langle Q \rangle = C(U_g - V_g), \tag{4}$$

where we have dropped constant terms. Since an electron in the dot behaves as a free particle subject to the effective potential U_g, the mean charge can be also expressed using the density of states in the dot $\mathrm{d}N/\mathrm{d}\varepsilon$ as

$$\langle Q \rangle = e \frac{\mathrm{d}N}{\mathrm{d}\varepsilon}(-eU_g) \equiv -C_q U_g. \tag{5}$$

C_q is called quantum capacitance since it reflects the nature of standing wave of an electron in the dot. By eliminating U_g from Eqs. (4) and (5), we can calculate the electrochemical capacitance

$$C_\mu = -\frac{\partial \langle Q \rangle}{\partial V_g} = \frac{C C_q}{C + C_q}, \qquad \therefore \frac{1}{C_\mu} = \frac{1}{C} + \frac{1}{C_q}. \tag{6}$$

Thus in the mean field approximation C_μ is reduced to a series combination of of the classical part C and the quantum part C_q, in agreement with the results of the scattering theory [4]. Note that $C_q \propto \mathrm{d}N/\mathrm{d}\varepsilon$ oscillates as V_g is varied. For an integer quantum Hall edge state at $\nu = 1$, one can refermionized the Hamiltonian to solve the transport problem with the effective potential U_g with the scattering theory. Then the charge relaxation resistance quantized at $R_q = h/(2e^2)$ will be also reproduced.

One might conjecture that the mean field assumption is valid in the limit of weak quasiparticle tunneling $V \to 0$, where the potential that dot charge Q feels is almost harmonic, so that the charge fluctuation is suppressed upon lowering temperature. Indeed, we can exactly derive quantized charge relaxation resistance $R_q = h/(2e^2\nu)$ up to second order in V [6], but a qualitative difference between the mean field theory and our exact result emerges in the capacitance. From the perturbative result [6], one can split C_μ into two parts mimicking Eq. (6):

$$\frac{1}{C_\mu} = \frac{1}{\gamma C} + \frac{1}{\gamma C_q'} \qquad \left(C_q' \equiv \frac{\nu^2 e^2 L}{2\pi v} \right), \tag{7}$$

where γ is a numerical factor oscillating as a function of V_g, and $L \equiv x_2 - x_1$ is the circumference of the dot. For $\nu = 1$, $\gamma C_q'$ can be identified as the quantum capacitance, since it coincides with the small V limit of C_q obtained from the scattering theory, while γC differs from the geometrical capacitance by the oscillating factor. The latter is because the mean field approximation underestimates the charge fluctuation ($\sim$ capacitance) due to electron tunneling between the dot and the reservoir, which should oscillate depending on V_g. Therefore, one can see from Eq. (7) that in the small V limit, C_μ is a series combination of γC and $\gamma C_q'$, which reflect the particle and wave natures of an electron, respectively.

It is clear that the mean field theory breaks down at charge degenerate points $\langle Q \rangle = (n+1/2)e$ (n is an integer) for large V, where the charge fluctuation can be finite down to low temperatures.

In Ref. [6], we have shown that the charge relaxation resistance for $\nu > 1/2$ is always quantized at $R_q = h/(2e^2\nu)$ owing to the so-called charge-Kondo effect [9], where two degenerate charge states ne and $(n+1)e$ play the role of up and down spins of a magnetic impurity [2]. More surprisingly, charge relaxation resistance becomes ill-defined due to the Kosterlitz-Thouless transition for $\nu < 1/2$, and the observed dynamical resistance R_q diverges with lowering temperature.

4. Noise-driven quantum phase transition

The phase transition mentioned in the previous section occurs for a fractional quantum Hall edge state such as $\nu = 1/3$. In the rest of this paper, we discuss another possibility for the phase transition by considering a noisy gate voltage connected to an ohmic bath. Such a situation can be realized, e.g., by connecting a long LC transmission line in series with the gate voltage [10]. The Hamiltonian describing the Caldeira-Leggett bath of an infinite number of harmonic oscillators reads [11]

$$H_B = \sum_j \left[\frac{P_j{}^2}{2M_j} + \frac{M_j \Omega_j{}^2}{2} \left(X_j - \frac{\lambda_j}{M_j \Omega_j{}^2} Q \right)^2 \right],$$ (8)

where λ_j is the strength of interaction between the jth oscillator and the dot charge Q. X_j, P_j, M_j, Ω_j denote the normal coordinate, momentum, mass, and frequency of the jth oscillator. For an ohmic bath with impedance R_B, the spectral density of the environmental coupling satisfies

$$J(\omega) \equiv \frac{\pi}{2} \sum_j \frac{\lambda_j{}^2}{M_j \Omega_j} \delta(\omega - \Omega_j) = R_B \omega.$$ (9)

Below, we see how dissipation influences the behavior of charge relaxation resistance. To this end, we derive the euclidian effective action for the system consisting of the mesoscopic capacitor and the ohmic bath. By integrating out the quadratic degrees of freedom except $\phi(x_1)$ and $\phi(x_2)$ from the kinetic term (1), we first obtain the effective action for the mesoscopic capacitor

$$S_{MC} = \frac{1}{\pi \nu \beta} \sum_{\omega_n} \frac{|\omega_n|}{1 - \mathrm{e}^{-|\omega_n| L / v}} |\tilde{\phi}(\omega_n)|^2 + V \int \mathrm{d}\tau \cos[2\phi(\tau)] + \int \mathrm{d}\tau \left[\frac{E_C}{\pi^2} \{\phi(\tau)\}^2 - \frac{eV_g}{\pi} \phi(\tau) \right].$$ (10)

Here $\phi \equiv [\phi(x_1) - \phi(x_2)]/2$, $\tilde{\phi}(\omega_n)$ is the Fourier transform of $\phi(\tau)$, and $E_C \equiv e^2/(2C)$ is the charging energy. Similarly, we can derive the effective action for the bath from Eq. (8) as

$$S_B = \frac{\alpha}{\pi \beta} \sum_{\omega_n} |\omega_n| |\tilde{\phi}(\omega_n)|^2 \qquad \left(\alpha = \frac{R_B}{h/e^2} \right),$$ (11)

where α is the strength of dissipation. One can expect from Eqs. (10) and (11) that the coupling to the ohmic bath renormalizes the filling factor ν, which influences the critical point. To discuss the quantum phase transition of the mesoscopic capacitor, let us focus on the degeneracy point in the large V region. In this case, electron tunneling occurs between the dot and the reservoir with small tunneling strength t. Following Ref. [6], we can straightforwardly identify the scaling equations at low frequencies $|\omega_n| \ll v/L$:

$$\frac{\mathrm{d}t}{\mathrm{d}l} = \left[1 - \left(\alpha + \frac{1}{2\nu} \right) s^2 \right] t, \qquad \frac{\mathrm{d}s^2}{\mathrm{d}l} = -4 s^2 t^2 \qquad (0 < s \lesssim 1),$$ (12)

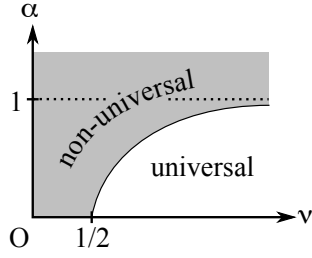[2] Recently, an equivalent Kondo physics at $\nu = 1$ has been also discussed in Ref. [7].

Figure 2. Phase diagram of the universal quantization of the charge relaxation resistance of the mesoscopic capacitor with a noisy gate voltage. ν is the filling factor, and $\alpha \equiv R_B/(h/e^2)$ is the strength of dissipation, where R_B is the bath impedance. The charge relaxation resistance is universally quantized in the white region, while it can undergoes a phase transition for sufficiently strong quasiparticle tunneling in the gray region.

which describes the Kosterlitz-Thouless transition. One can see from these equations that, for $\alpha < 1 - 1/(2\nu)$, the tunneling strength always grows upon decreasing temperature. For $\alpha > 1 - 1/(2\nu)$, on the other hand, it is possible that for sufficiently large V (small t) that the system is renormalized to a weak coupling configuration with specified charge. It should be noted that, unlike in Ref. [6], the critical phase transition occurs even in the case of an integer quantum Hall edge ($\nu = 1$) when the bath impedance exceeds half a resistance quantum, $R_B > h/(2e^2)$. Otherwise, the system flows towards the Kondo fixed point as $T \to 0$, and the charge relaxation resistance is given by the value for $V \to 0$ as $R_q = h/(2e^2) + R_B$, i.e., the noisy gate voltage results in the shift of the quantized value by the bath impedance R_B. The phase diagram of the universal quantization of the charge relaxation resistance is shown in Fig. 2.

5. Summary
We have theoretically studied the phase transition of dynamical resistance in a mesoscopic capacitor applied a perpendicular magnetic field. For weak quasiparticle tunneling, the electrochemical capacitance can be expressed as a series combination of two capacitors, one of which shows a behavior qualitatively different from the result of the scattering theory. We have also discussed the phase transition driven by noise with with dissipation strength α. When $\alpha < 1 - 1/(2\nu)$, the charge relaxation resistance is universally quantized at $h/(2e^2\nu) + R_B$, with the bath impedance R_B.

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